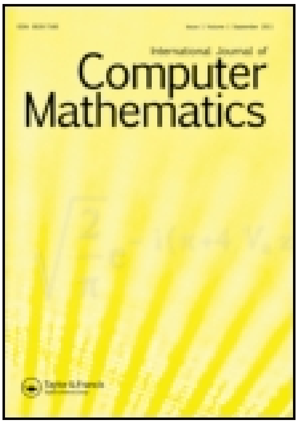

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Crown graphs and subdivision of ladders are odd graceful

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A graph G of size q is odd graceful, if there is an injection ϕ from $V(G)$ to $\{0, 1, 2, \ldots, 2q-1\}$ such that, when each edge xy is assigned the label or weight $|f(x) - f(y)|$, the resulting edge labels are $\{1, 3, 5, \ldots, 2q-1\}$. This definition was introduced in 1991 by Gnanajothi [3], who proved that the graphs obtained by joining a single pendant edge to each vertex of C_n are odd graceful, if n is even. In this paper, we generalize Gnanajothi's result on cycles by showing that the graphs obtained by joining m pendant edges to each vertex of C_n are odd graceful if n is even. We also prove that the subdivision of ladders $S(L_n)$ (the graphs obtained by subdividing every edge of L_n exactly once) is odd graceful.

Keywords: odd graceful; crown graphs; ladder graphs

2010 AMS Subject Classification: 05C78

1. Introduction

The study of graceful graphs and graceful labelling methods was introduced by Rosa [6]. Rosa defined a β-valuation of a graph G with q edges as an injection from the vertices of G to the set $\{0, 1, 2, \ldots, q\}$ such that when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. β-Valuations are the functions that produce graceful labellings. However, the term graceful labelling was not used until Golomb studied such labellings several years later [4]. The notation of graceful labelling was introduced as a tool for decomposing the complete graph into isomorphic subgraphs. Graph labellings can also be applied in the areas such as coding theory, communication networks, mobile telecommunications or optimal circuits layouts.

Many of the results about graph labelling are collected and updated regularly in a survey by Gallian [2]. The reader can consult this survey for more information about the subject.

A graph G of size q is odd graceful, if there is an injection f from $V(G)$ to $\{0, 1, 2, \ldots, 2q-1\}$ such that, when each edge xy is assigned the label or weight $|f(x) - f(y)|$, the resulting edge labels are $\{1, 3, 5, \ldots, 2q-1\}$. This definition was introduced by Gnanajothi [3] in 1991, who proved that every cycle C_n is odd graceful if n is even. We denote the crown graphs (the graphs obtained by joining a single pendant edge to each vertex of C_n) by $C_n \odot K_1$; therefore, the crown graphs

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(the graphs obtained by joining m pendant edges to each vertex of C_n,) are denoted by $C_n \odot mK_1$. Gnanajothi [3] proved that $C_n \odot K_1$ are odd graceful if n is even. In our study, we generalize Gnanajothi's result on cycles by showing that the graphs $(C_n \odot mK_1)$ are odd graceful if n is even. We also prove that the subdivision of ladders $S(L_n)$ (the graphs obtained by subdividing every edge of L_n exactly once) is odd graceful. In Section 2, we prove that the crown graphs $C_n \odot mK_1$ are odd graceful. The odd graceful subdivision of ladders $S(L_n)$ will be introduced in Section 3.

2. Odd graceful labellings of crown graphs

We are introducing a special method for labelling the vertices of the cyclic graph. First, we show how to use this method in order to prove that the cycle C_n is odd graceful. Second, we use this method for proving further theorems. We denote the total number of edges of the crown graphs $C_n \odot mK_1$ by q.

THEOREM 2.1 *The cycle C_n is odd graceful if n is even* ($n \geq 4$).

Proof We can see a cycle as the graph which consists of two paths, a left path $uu_1u_2u_3 \cdots u_{(n/2)-1}$ and a right path $v_1v_2v_3 \cdots v_{n/2} = v$. In order to get the cycle, connect the vertex u with the vertex v_1 and connect the vertex $v_{n/2}$ with the vertex $u_{(n/2)-1}$ (Figure 1(a)). Clearly, C_n has n vertices and q edges such that $q = n$. Let us consider the following numbering ϕ of the vertices of C_n.

$$\begin{aligned}
&\phi(u) = 0,\\
&\phi(u_i) = q - i \quad (i \text{ odd}), \quad i = 1, 2, \ldots, \left(\tfrac{n}{2}\right) - 1,\\
&\phi(u_i) = q + i \quad (i \text{ even}), \quad i = 1, 2, \ldots, \left(\tfrac{n}{2}\right) - 1,\\
&\phi(v_i) = 2q - i \quad (i \text{ odd}), \quad i = 1, 2, \ldots, \left(\tfrac{n}{2}\right) - 1, \tfrac{n}{2},\\
&\phi(v_i) = i \quad (i \text{ even}), \quad i = 1, 2, \ldots, \left(\tfrac{n}{2}\right) - 1, \tfrac{n}{2}.
\end{aligned}$$

(1)

$$\begin{aligned}
\max_{v \in V} \phi(v) &= \max\left\{0, \max_{\substack{i \text{ odd}\\1 \leq i \leq (n/2)-1}} q - i, \max_{\substack{i \text{ even}\\1 \leq i \leq (n/2)-1}} q + i, \max_{\substack{i \text{ odd}\\1 \leq i \leq (n/2)}} 2q - i, \max_{\substack{i \text{ even}\\1 \leq i \leq (n/2)}} i\right\}\\
&= 2q - 1, \text{ the maximal value of all odds.}
\end{aligned}$$

Thus, $\phi(v) \in \{0, 1, 2, \ldots, 2q - 1\}$.

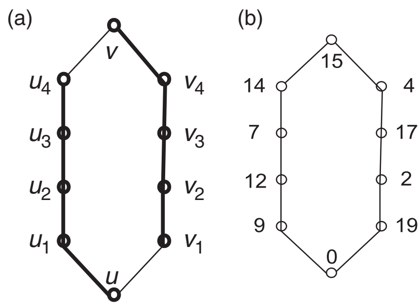

Figure 1. C_{10} is odd graceful.

(2) Clearly, ϕ is a one-to-one mapping from $V(C_n)$ to $\{0, 1, 2, \ldots, 2q-1\}$.

(3) It remains to show that the labels of the edges of C_n are the odd integers of the set $\{1, 3, 5, \ldots, 2q-1\}$.

The range of $|\phi(v_i) - \phi(v_{i+1})| = \{2q - 2i - 1 : i = 1, 2, 3, \ldots, (\frac{n}{2}) - 1\} = \{2q-3, 2q-5, \ldots, q+1\}$.

The range of $|\phi(u_i) - \phi(u_{i+1})| = \{2i + 1 : i = 1, 2, 3, \ldots, (\frac{n}{2}) - 2\} = \{3, 5, \ldots, q-3\}$.

The edges uu_1, uv_1 and $vu_{(n/2)-1}$ are labelled by $q-1, 2q-1$ and 1, respectively. Hence, $|\{\phi(u) - \phi(v)| : uv \in E\} = \{1, 3, 5, \ldots, 2q-1\}$, so that C_n is odd graceful if n is even. ■

THEOREM 2.2 *The crown graph $C_n \odot mK_1$ is odd graceful if n is even* ($n \geq 4$).

Proof We can see a cycle as the graph which consists of two paths, a left path $uu_1u_2u_3 \ldots, u_{(n/2)-1}$ and a right path $v_1v_2v_3 \ldots, v_{n/2}$. In order to get the cycle, we connect the vertex u with the vertex v_1 and connect the vertex $v_{n/2}$ with the vertex $u_{(n/2)-1}$. Now, we can get the crown graphs $C_n \odot mK_1$ by joining m pendant edges for each vertex v_i on the right path $(v_iv_i^1, v_iv_i^2, \ldots, v_iv_i^m)$, where $i = 1, 2, 3, \ldots, \frac{n}{2}$ and joining m pendant edges for each vertex u_i on the left path $(u_iu_i^1, u_iu_i^2, \ldots, u_iu_i^m)$, where $i = 1, 2, 3, \ldots, (\frac{n}{2}) - 1$. Finally, we join m pendant edges for the vertex u $(uu^1, uu^2, \ldots, uu^m)$ (Figure 2). Clearly, $C_n \odot mK_1$ has $n + nm$ vertices and $n + nm$ edges such that $q = n + nm$. We assume that the number of vertices $u_1u_2u_3 \cdots u_{(n/2)-1}$ and the number of vertices $v_1v_2v_3 \cdots v_{(n/2)-1}$ are both equal to x.

Case 1 (If x is even). Let us consider the following numbering ϕ of the vertices of $C_n \odot mK_1$.

$$\phi(u) = 0,$$

$$\phi(u_i) = q + (m+1)(i-1) - 1 \quad (i \text{ odd}), \quad i = 1, 2, \ldots, \left(\tfrac{n}{2}\right) - 1,$$

$$\phi(u_i) = 2q - (m+1)(i+1) + (m-1) \quad (i \text{ even}), \quad i = 1, 2, \ldots, \left(\tfrac{n}{2}\right) - 1,$$

$$\phi(u_i^k) = 2q - (m+1)(i+1) - 2k + 2m \quad (i \text{ odd}), \quad k = 1, 2, \ldots, m\ i = 1, 2, \ldots, \left(\tfrac{n}{2}\right) - 1,$$

$$\phi(u_i^k) = q + (m+1)(i-1) + 2k - m - 2 \quad (i \text{ even}), \quad k = 1, 2, \ldots, m\ i = 1, 2, \ldots, \left(\tfrac{n}{2}\right) - 1,$$

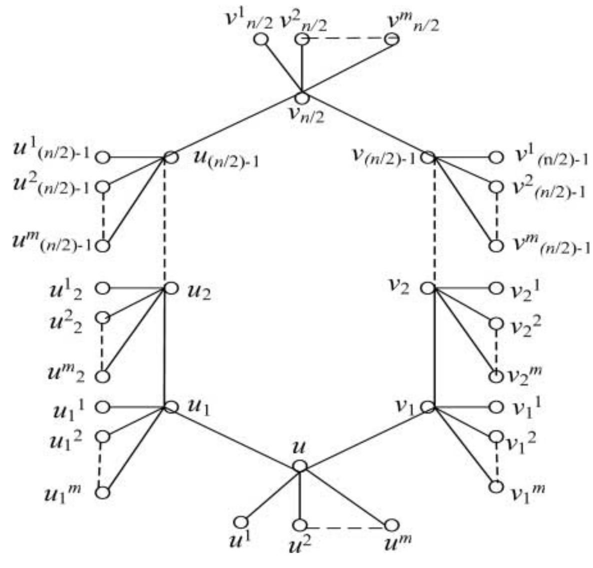

Figure 2. $C_n \odot mK_1$ is odd graceful.

$\phi(u^k) = 2k+1, k = 1,2,\ldots,m,$

$\phi(v_i) = 2q-(m+1)(i-1)-1 \quad (i \text{ odd}), \quad i = 1,2,\ldots,\left(\frac{n}{2}\right)-1, \frac{n}{2},$

$\phi(v_i) = (m+1)i \quad (i \text{ even}), \quad i = 1,2,\ldots,\left(\frac{n}{2}\right)-1, \frac{n}{2},$

$\phi(v_i^k) = (m+1)(i-1)+2k \quad (i \text{ odd}), \quad k = 1,2,\ldots,m,\ i = 1,2,\ldots,\left(\frac{n}{2}\right)-1, \frac{n}{2},$

$\phi(v_i^k) = 2q-(m+1)(i-1)-2k+m \quad (i \text{ even}), \quad k = 1,2,\ldots,m,\ i = 1,2,\ldots,\left(\frac{n}{2}\right)-1, \frac{n}{2}.$

(1)

$$\begin{aligned}\max_{v\in V}\\ \phi(v) = \max\Bigg\{0, &\overset{i\text{ odd}}{\max_{1\le i\le (n/2)-1}} q+(m+1)(i-1)-1, \overset{i\text{ even}}{\max_{1\le i\le n/2}}(m+1)i,\\ &\overset{i\text{ even}}{\max_{1\le i\le (n/2)-1}} 2q-(m+1)(i+1)+(m-1),\\ &\overset{1\le k\le m,\ i\text{ odd}}{\max_{1\le i\le (n/2)-1}} 2q-(m+1)(i+1)-2k+2m,\\ &\overset{1\le k\le m,\ i\text{ even}}{\max_{1\le i\le (n/2)-1}} q+(m+1)(i-1)+2k-m-2,\\ &\overset{1\le k\le m}{\max} 2k+1, \quad \overset{i\text{ odd}}{\max_{1\le i\le n/2}} 2q-(m+1)(i-1)-1,\\ &\overset{1\le k\le m,\ i\text{ odd}}{\max_{1\le i\le n/2}}(m+1)(i-1)+2k,\\ &\overset{1\le k\le m,\ i\text{ even}}{\max_{1\le i\le n/2}} 2q-(m+1)(i-1)-2k+m\Bigg\}\\ = 2q-1, &\quad \text{the maximal value of all odds.}\end{aligned}$$

Thus, $\phi(v) \in \{0,1,2,\ldots,2q-1\}$.

(2) Clearly, ϕ is a one-to-one mapping from $V(C_n \odot mK_1)$ to $\{0,1,2,\ldots,2q-1\}$.

(1) It remains to show that the labels of the edges of $C_n \odot mK_1$ are all the odd integers of the set $\{1,3,5,\ldots,2q-1\}$.
The range of $|\phi(u_i)-\phi(u_{i+1})| = \{q-2i(m+1)-1;\quad i = 1,2,3,\ldots,(\frac{n}{2})-2\} = \{q-2m-3, q-4m-5,\ldots,q-(n-4)(m+1)-1\}$.
The range of $|\phi(v_i)-\phi(v_{i+1})| = \{2q-2i(m+1)-1;\ i = 1,2,3,\ldots,(\frac{n}{2})-1\} = \{2(q-m)-3, 2(q-2m)-5,\ldots,2q-(n-2)(m+1)-1\}$.
The range of $|\phi(v_i)-\phi(v_i^k)| = \{2q-2(i-1)(m+1)-2k-1 \quad (i \text{ odd}),$ $k = 1,2,3,\ldots,m; \quad i = 1,2,3,\ldots,\frac{n}{2}\} = \{2(q-k)-1, 2(q-k)-9,\ldots,2(q-k)-(n-2)(m+1)-1\}$.
The range of $|\phi(v_i)-\phi(v_i^k)| = \{2(q-k)-(m+1)(2i-1)+m \quad (i \text{ even}), \quad k = 1,2,3,\ldots,m; i = 1,2,3,\ldots,\frac{n}{2}\} = \{2(q-k)-2m-3, 2(q-k)-6m-7,\ldots,2(q-k)-n(m+1)+2m+1\}$.
The range of $|\phi(u_i)-\phi(u_i^k)| = \{q-2(mi+i+k-m)+1 \quad (i \text{ odd}), \quad k = 1,2,3,\ldots,m;$ $i = 1,2,3,\ldots,(\frac{n}{2})-1\} = \{q-2k-1, q-4m-2k-5,\ldots,q-mn+4m-n-2k+3\}$.
The range of $|\phi(u_i)-\phi(u_i^k)| = \{q-2(mi+i+k-1)+m \quad (i \text{ even}), \quad k = 1,2,3,\ldots,m;$ $i = 1,2,3,\ldots,(\frac{n}{2})-1\} = \{q-3m-2k-2, q-7m-2k-6,\ldots,q-mn+3m-n-2k+4\}$.
The range of $|\phi(u)-\phi(u^k)| = \{2k+1;\ k = 1,2,3,\ldots,m\} = \{3,5,\ldots,2m+1\}$.

The edges uu_1, uv_1 and $vu_{(n/2)-1}$ are labelling by $q-1, 2q-1$ and 1, respectively. Hence, $|\{\phi(u)-\phi(v)|: uv \in E\} = \{1,3,5,\ldots,2q-1\}$ so that $C_n \odot mK_1$ is odd graceful if n is even.

Case 2 (If x is odd). Let us consider the following numbering ϕ of the vertices of $C_n \odot mK_1$.

$\phi(u) = 0,$

$\phi(u_i) = q + (m+1)(i+1) - 1 \quad (i \text{ odd}), \quad i = 1,2,\ldots,(\frac{n}{2})-1,$

$\phi(u_i) = q - (m+1)(i-1) - (m-1) \quad (i \text{ even}), \quad i = 1,2,\ldots,(\frac{n}{2})-1,$

$\phi(u_i^k) = q - (m+1)(i-1) - 2k + 2 \quad (i \text{ odd}); \quad k = 1,2,\ldots,m;\ i = 1,2,\ldots,(\frac{n}{2})-1,$

$\phi(u_i^k) = q + (m+1)(i+1) + 2k - m - 2 \quad (i \text{ even}), \quad k = 1,2,\ldots,m;\ i = 1,2,\ldots,(\frac{n}{2})-1,$

$\phi(u^k) = 2k - 1 \quad k = 1,2,\ldots,m,$

$\phi(v_i) = 2q - (m+1)(i-1) - 1 \quad (i \text{ odd}), \quad i = 1,2,\ldots,(\frac{n}{2})-1, \frac{n}{2},$

$\phi(v_i) = (m+1)i \quad (i \text{ even}), \quad i = 1,2,\ldots,(\frac{n}{2})-1, \frac{n}{2},$

$\phi(v_{n/2}) = m(x+1) + x + 3,$

$\phi(v_i^k) = (m+1)(i-1) + 2k \quad (i \text{ odd}), \quad k = 1,2,\ldots,m;\ i = 1,2,\ldots,(\frac{n}{2})-1, \frac{n}{2},$

$\phi(v_i^k) = 2q - (m+1)(i-1) - 2k + m \quad (i \text{ even}), \quad k = 1,2,\ldots,m;\ i = 1,2,\ldots,\frac{n}{2}-1, \frac{n}{2}.$

(1)

$$\begin{aligned}
\max_{v\in V} \phi(v) = \max \Bigg\{ 0, \ & \max_{1\le i\le (n/2)-1}^{i \text{ odd}} q + (m+1)(i+1) - 1, \\
& \max_{1\le i\le (n/2)-1}^{i \text{ even}} q - (m+1)(i-1) - (m-1), \\
& \max_{1\le i\le (n/2)-1}^{1\le k\le m,\ i \text{ odd}} q - (m+1)(i-1) - 2k + 2, \\
& \max_{1\le i\le (n/2)-1}^{1\le k\le m,\ i \text{ even}} q + (m+1)(i+1) + 2k - m - 2, \ \max^{1\le k\le m} 2k - 1, \\
& \max_{1\le i\le n/2}^{i \text{ odd}} 2q - (m+1)(i-1) - 1, \ \max_{1\le i\le n/2}^{i \text{ even}} (m+1)i, \\
& \max_{1\le i\le n/2}^{1\le k\le m,\ i \text{ odd}} (m+1)(i-1) + 2k, \max m(x+1) + x + 3, \\
& \max_{1\le i\le n/2}^{1\le k\le m,\ i \text{ even}} 2q - (m+1)(i-1) - 2k + m \Bigg\} \\
= 2q - 1, & \text{ the maximal value of all odds.}
\end{aligned}$$

Thus, $\phi(v) \in \{0,1,2,\ldots,2q-1\}$.

(2) Clearly, ϕ is a one-to-one mapping from $V(C_n \odot mK_1)$ to $\{0,1,2,\ldots,2q-1\}$.

(1) It remains to show that the labels of the edges of $C_n \odot mK_1$ are all the odd integers of the set $\{1,3,5,\ldots,2q-1\}$.
The range of $|\phi(u_i) - \phi(u_{i+1})| = \{2(m + im + i) - 1; i = 1,2,\ldots,(\frac{n}{2})-2\}$

$= \{4m+1, 6m+3, \dots, nm-2m+n-3\}$.
The range of $|\phi(v_i) - \phi(v_{i+1})| = \{2q - 2i(m+1) - 1; i = 1,2,3,\dots,(\frac{n}{2}) - 1\} = \{2(q-m) - 3, 2(q-2m) - 5, \dots, 2q - (n-2)(m+1) - 1\}$.
The range of $|\phi(v_i) - \phi(v_i^k)| = \{2(q-k) - 2(m+1)(i-1) - 1(i \text{ odd}), k = 1,2,3,\dots,m; i = 1,2,3,\dots,\frac{n}{2}\} = \{2(q-k) - 1, 2(q-k) - 4m - 5, \dots, 2(q-k) - n(m+1) + 2m + 1\}$.
The range of $|\phi(v_i) - \phi(v_i^k)| = \{2(q-k) - (m+1)(2i-1) + m; (i \text{ even}); k = 1,2,3,\dots, m; i = 1,2,3,\dots,\frac{n}{2}\} = \{2(q-k) - 2m - 3, 2(q-k) - 6m - 7, \dots, 2(q-k) - n(m+1) + 2m + 1\}$.
The range of $|\phi(u_i) - \phi(u_i^k)| = \{2i(m+1) + 2k - 3 \quad (i \quad \text{odd}), k = 1,2,3,\dots,m; i = 1,2,3,\dots,(\frac{n}{2}) - 1\} = \{2(m+k+1), 6m+2k+3, \dots, n(m+1) - 2m + 2k - 5\}$.
The range of $|\phi(u_i) - \phi(u_i^k)| = \{2i(m+1) + 2k - 3 \ (i \ \text{even}), k = 1,2,3,\dots,m; \quad i = 1,2,3,\dots,(\frac{n}{2}) - 1\} = \{4m+2k+1, 8m+2k+5, \dots, (m+1)(n-2) + 2k - 3\}$.
The range of $|\phi(u) - \phi(u^k)| = \{2k-1; k = 1,2,3,\dots,m\} = \{1,3,5,\dots,2m-1\}$.
The edges uu_1, uv_1 and $vu_{(n/2)-1}$ are labelling by $q + 2(m+1) - 1, 2q - 1$ and $q - 3$, respectively. Hence, $|\{\phi(u) - \phi(v)| : \quad uv \in E\} = \{1,3,5,\dots,2q-1\}$ so that $C_n \odot mK_1$ is odd graceful if n is even. Therefore, $C_n \odot mK_1$ is odd graceful if n is even. ■

3. Odd graceful of subdivision of ladders

The ladder graph L_n is defined by $L_n = P_n \times K_2$, where P_n is a path with n vertices and $\times$ denotes the Cartesian product (L_n has $2n$ vertices). Bodendiek *et al.* [1] proved that L_n is graceful. Maheo [5] proved that L_n is strongly graceful. In this section, we prove that the subdivision of graphs $S(L_n)$ (obtained by subdividing every edge of L_n exactly once) is odd graceful.

THEOREM 3.1 *The subdivision of ladders $S(L_n)$ is odd graceful.*

Proof Let $u_1u_2u_3 \cdots u_n, v_1v_2v_3 \cdots v_n$ be the vertices of the ladder L_n. Let v_i' be the newly added vertex between v_i and v_{i+1}, u_i' be the newly added vertex between u_i and u_{i+1} and w_i be the newly added vertex between u_i and v_i, (Figure 3). Clearly, $S(L_n)$ has $5n-2$ vertices and $6n-4$ edges such that $q = 6n-4$. Let us consider the following numbering ϕ of the vertices of $S(L_n)$.

$$\phi(u_i) = q + 2i - 3, \quad i = 1,2,\dots,n.$$

$$\phi(v_i) = 2q - 2i + 1, \quad i = 1,2,\dots,n.$$

$$\phi(u_i') = 2q - 2i - 2n, \quad i = 1,2,\dots,n-1.$$

$$\phi(v_i') = 4i - 2, \quad i = 1,2,\dots,n-1.$$

$$\phi(w_i) = 4i - 4, \quad i = 1,2,\dots,n.$$

(1)

$$\max_{v \in V} \phi(v) = \max \left\{0, \max_{1 \le i \le n} q + 2i - 3, \max_{1 \le i \le n} 2q - 2i + 1, \max_{1 \le i \le n-1} 2q - 2i - 2n, \right.$$
$$\left. \max_{1 \le i \le n-1} 4i - 2, \quad \max_{1 \le i \le n} 4i - 4 \right\}$$
$$= 2q - 1, \quad \text{the maximal value of all odds.}$$

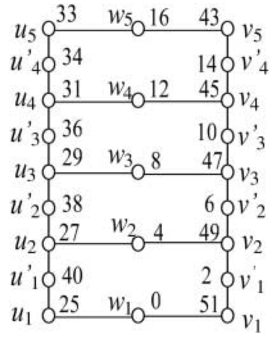

Figure 3. $S(L_5)$ is odd graceful.

Thus, $\phi(v) \in \{0, 1, 2, \ldots, 2q-1\}$.

(2) Clearly, ϕ is a one-to-one mapping from $V(S(L_n))$ to $\{0, 1, 2, \ldots, 2q-1\}$.

(3) It remains to show that the labels of the edges of $S(L_n)$ are all the odd integers of the set $\{1, 3, 5, \ldots, 2q-1\}$.
The range of $|\phi(u_i{}') - \phi(u_i)| = \{q - 2n - 4i + 3;\ i = 1, 2, 3, \ldots, n\}$
$= \{q - 2n - 1, q - 2n - 5, \ldots, q - 6n + 7\}$.
The range of $|\phi(u_i) - \phi(u'_{i-1})| = \{q - 2n - 4i + 5;\ i = 1, 2, 3, \ldots, n\}$
$= \{q - 2n - 3, q - 2n - 7, \ldots, q - n + 5\}$.
Similarly,
the range of $|\phi(u_i) - \phi(w_i)| = \{q-1, q-3, q-5, \ldots, q-2n+3, q-2n+1\}$,
the range of $|\phi(v_i) - \phi(w_i)| = \{2q-1, 2q-7, q-5, \ldots, 2q-6n+11, 2q-6n+5\}$,
the range of $|\phi(v'_i) - \phi(v_i)| = \{2q-3, 2q-9, \ldots, 2q-6n+15, 2q-6n+3\}$,
the range of $|\phi(v_i) - \phi(v'_{i-1})| = \{2q-5, 2q-11, \ldots, 2q-6n+13, 2q-6n+7\}$.
Hence, $|\{\phi(u) - \phi(v)| : \quad uv \in E\} = \{1, 3, 5, \ldots, 2q-1\}$ so that $S(L_n)$ is odd graceful for all n. ■

4. Conclusion

Since labelled graphs serve as practically useful models for wide-ranging applications such as communications network, circuit design, coding theory, radar, astronomy, X-ray and crystallography, it is desired to have generalized results or results for a whole class, if possible. This work has presented the generalized result to obtain the graphs (obtained by joining m pendant edges to each vertex of C_n) which are odd graceful if n is even. Finally, we have proved that the subdivision of ladders $S(L_n)$ is odd graceful.

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